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# THE CONJUGATION-POINT AND THE APPLICATION OF ITS PROPERTIES TO GRAPHIC CONSTRUCTIONS.

By PROF. FRANK H. LOUD, Colorado Springs, Colo.

The theorem which I propose to state, and to apply to some simple problems of the geometry of the conic, is itself readily deduced from principles relating to involutions, well known to every student of the projective geometry. But in this form it is new, so far as I am aware; and as I think its practical applications may render it serviceable to teachers of elementary analytical geometry, I shall state and prove it in terms adapted to classes in that branch.

*Definition.*—If through the centre of a given conic there be drawn (1) the circumference of a circle, (2) a tangent to that circumference, which tangent will of course be a diameter of the conic, (3) a second diameter, conjugate to the one just mentioned,—then the point on this second diameter where it is intersected by a line joining those two points on the circumference of the circle in which the latter meets the two axes of the conic separately is called the *conjugation-point* of the conic in respect to the circle.

*To determine the coordinates of the conjugation-point.*—The definition suggests the convenient method of finding these, which would doubtless be taken unaided by the elementary student. If the conic be given by the equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , and the coordinates of the centre of the circle be  $m, n$ , then the circle and lines named in the definition have successively the equations,  $x^2 - 2mx + y^2 - 2ny = 0$ ,  $mx + ny = 0$ ,  $b^2nx - a^2my = 0$ , and  $nx + my - 2mn = 0$ . By elimination from the last two the required coordinates are found  $x = \frac{2a^2m}{a^2 + b^2}$ ,  $y = \frac{2b^2n}{a^2 + b^2}$ . To adapt the formula to the hyperbola requires of course a change of the sign of  $b^2$ .

*Theorem.*—The conjugation-point of a conic with respect to a circle is in the line joining the two points in which the circle is cut by any pair of conjugate diameters of the conic separately.

To prove this theorem, using the formula just obtained, we have first to assume an equation to represent any diameter, which, for the sake of symmetry, may be taken to be  $bsx - ay = 0$  then that of the conjugate will be  $bx + asy = 0$ . Then the two points which we have to show to be in one line with the point  $\frac{2a^2m}{a^2 + b^2}, \frac{2b^2n}{a^2 + b^2}$ , are found to be  $\frac{2a^2m + 2abns}{a^2 + b^2s^2}, \frac{2abms + 2b^2ns^2}{a^2 + b^2s^2}$  and  $\frac{2a^2ms^2 - 2abns}{a^2s^2 + b^2}, \frac{-2abms + 2b^2n}{a^2s^2 + b^2}$ . If, now, this must be shown by ele-

mentary algebraic methods, using the formula of the common text-books, the process is indeed somewhat tedious, but of course in no other sense difficult. But if determinants may be used, the proof is short enough. For, in this case, the formula

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

readily takes the form

$$\frac{4ab}{(a^2 + b^2)(a^2 + b^2s^2)(a^2s^2 + b^2)} \begin{vmatrix} am & bn & a^2 + b \\ am + bns & ams + bns^2 & a^2 + b^2s^2 \\ ams^2 - bns & -ams + bn & a^2s^2 + b^2 \end{vmatrix}.$$

Now, if for the second or third row be substituted the sum of these two, divided through by  $1 + s^2$ , the determinant is found to have two rows identical.

*Problem.*—Having given in position, (not in length), two diameters of a conic and the conjugate of each, to construct, in position, the conjugate of any other given diameter, the axes of the curve, and (if the latter be an hyperbola) the asymptotes. (The conic itself is supposed not to be drawn.)

Draw, through the intersection of the given diameters, the circumference of any convenient circle, producing, if necessary, all the diameters to meet the latter. Join the point in which any diameter meets the circumference with that in which its conjugate meets the same; two lines are thus found, on each of which lies the conjugation-point; the latter is therefore at their intersection. Join the point in which the fifth given diameter meets the circle with the conjugation-point, and the point in which this line meets the circle again with the centre of the conic; this line is the required conjugate diameter. (It is on account of the facility with which a conjugate is found to a given diameter that I have suggested the name *conjugation-point*.) To locate the axes, join the centre of the circle to the conjugation-point; the points in which this line cuts the circumference are on the axes respectively. For the lines from these points to the centre are conjugate diameters as above; they also make an angle which is right, because inscribed in a semicircle. If the conic whose diameters are given is an ellipse, the conjugation-point will be found within the circle of construction; if it be an hyperbola, without. From this point draw tangents to the circle, and connect the points of contact with the centre of the conic; these lines are the asymptotes. For each is a diameter conjugate to itself.

The foregoing is sufficient to show the convenience with which this point may be used in constructions.